

GEOMETRIC REALIZATIONS OF GENERALIZED ALGEBRAIC CURVATURE OPERATORS

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ABSTRACT. We study the 8 natural GL equivariant geometric realization questions for the space of generalized algebraic curvature tensors. All but one of them is solvable; a non-zero projectively flat Ricci antisymmetric generalized algebraic curvature is not geometrically realizable by a projectively flat Ricci antisymmetric torsion free connection.

1. INTRODUCTION

The Ricci tensor and the Weyl projective curvature operator have always been important in mathematical physics. They play a central role in our analysis. Projective structures are of particular interest in both mathematics and in mathematical physics. Weyl [23] used projective structures to attempt a unification of gravitation and electro magnetics by constructing a model of space-time geometry combining both structures. His particular approach failed for physical reasons but his model is still studied; see [5, 8, 17, 9]. More recently, as the field is a vast one, we refer to a few recent references [4, 6, 10, 11, 13, 15, 16] to give a flavor of the context in which these concepts appear.

Let M be a smooth manifold of dimension m . We shall assume $m \geq 3$ henceforth to avoid complicating the exposition unduly as the 2-dimensional setting is a bit different. Let ∇ be a torsion free connection on the tangent bundle TM and let

$$\mathcal{R}^\nabla(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]}$$

be the curvature operator; this $(3, 1)$ tensor satisfies the identities:

$$(1.a) \quad \mathcal{R}^\nabla(x, y) = -\mathcal{R}^\nabla(y, x),$$

$$(1.b) \quad \mathcal{R}^\nabla(x, y)z + \mathcal{R}^\nabla(y, z)x + \mathcal{R}^\nabla(z, x)y = 0.$$

The relation of Equation (1.b) is called the *first Bianchi identity*. If $P \in M$, we let $\mathcal{R}_P^\nabla \in \otimes^3 T_P^* M \otimes T_P M = \otimes^2 T_P^* M \otimes \text{End}(T_P M)$ be the restriction of \mathcal{R}^∇ to $T_P M$.

It is convenient to pass to a purely algebraic context. Let V be a vector space of dimension m . A tensor $\mathcal{A} \in \otimes^2 V^* \otimes \text{End}(V)$ satisfying the symmetries given in Equations (1.a) and (1.b) is called a *generalized algebraic curvature operator* and we let $\mathfrak{A}(V) \subset \otimes^2 V^* \otimes \text{End}(V)$ be the subspace of all such operators. The fundamental question that we shall be examining in this paper is the extent to which algebraic properties of \mathcal{A} can be realized geometrically. Since we are working locally, we may assume without loss of generality that $M = V$.

One has the following result; although this result is well known, we shall give the proof in Section 2 for the sake of completeness as it is relatively short and as it contains a basic construction that is fundamental to our later results.

Theorem 1.1. *Let $\mathcal{A} \in \mathfrak{A}(V)$. There exists a torsion free connection ∇ on TV so that $\mathcal{R}_0^\nabla = \mathcal{A}$.*

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There are other geometric properties it is natural to study and which are invariant under the action of the general linear group $\mathrm{GL}(V)$, i.e. which do not depend on the choice of a basis for V . In either the algebraic or the geometric setting, one defines the *Ricci tensor* $\rho(\mathcal{A}) \in \otimes^2 V^*$ by setting

$$\rho(\mathcal{A})(x, y) := \mathrm{Tr}\{z \rightarrow \mathcal{A}(z, x)y\}.$$

Decompose $\rho(\mathcal{A}) = \rho_s(\mathcal{A}) + \rho_a(\mathcal{A})$ where $\rho_s(\mathcal{A}) \in S^2(V^*)$ is a symmetric bilinear form and where $\rho_a(\mathcal{A}) \in \Lambda^2(V^*)$ is an antisymmetric bilinear form by setting

$$\begin{aligned} \rho_s(\mathcal{A})(x, y) &:= \frac{1}{2}\{\rho(\mathcal{A})(x, y) + \rho(\mathcal{A})(y, x)\}, \\ \rho_a(\mathcal{A})(x, y) &:= \frac{1}{2}\{\rho(\mathcal{A})(x, y) - \rho(\mathcal{A})(y, x)\}. \end{aligned}$$

Definition 1.2. Let $\mathcal{A} \in \mathfrak{A}(V)$.

- (1) \mathcal{A} is Ricci symmetric if and only if $\rho(\mathcal{A}) \in S^2(V^*)$ i.e. $\rho_a(\mathcal{A}) = 0$.
- (2) \mathcal{A} is Ricci antisymmetric if and only if $\rho(\mathcal{A}) \in \Lambda^2(V^*)$ i.e. $\rho_s(\mathcal{A}) = 0$.
- (3) \mathcal{A} is Ricci flat if and only if $\rho(\mathcal{A}) = 0$.

We say a connection ∇ is *Ricci symmetric* if the associated Ricci tensor is symmetric; such connections are also often called *equiaffine connections*; they play a central role in many settings – see, for example, the discussion in [1, 3, 12, 14, 18]. Although the following result is well known [19], we present the proof in Section 3 since again the proof is short and the constructions involved play a crucial role in our development. If (\mathcal{O}, x) is a system of local coordinates on M , let

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \Gamma_{ij}^k \partial_{x_k}$$

define the *Christoffel symbols* of ∇ ; we adopt the *Einstein convention* and sum over repeated indices. Set

$$\omega_{\mathcal{O}} := \Gamma_{ij}^j dx^i.$$

Theorem 1.3. *Let ∇ be a torsion free connection. The following assertions are equivalent:*

- (1) *One has that $d\omega_{\mathcal{O}} = 0$ for any system of local coordinates \mathcal{O} on M .*
- (2) *One has that $\mathrm{Tr}(\mathcal{R}^\nabla) = 0$, i.e. \mathcal{R}^∇ is trace free.*
- (3) *The connection ∇ is Ricci symmetric.*
- (4) *The connection ∇ locally admits a parallel volume form.*

We will establish the following geometric realizability result in Section 4:

Theorem 1.4. *Let $\mathcal{A} \in \mathfrak{A}(V)$. Then:*

- (1) *If \mathcal{A} is Ricci symmetric, there exists a Ricci symmetric connection ∇ on TV so $\mathcal{R}_0^\nabla = \mathcal{A}$.*
- (2) *If \mathcal{A} is Ricci antisymmetric, there exists a Ricci antisymmetric connection ∇ on TV so $\mathcal{R}_0^\nabla = \mathcal{A}$.*
- (3) *If \mathcal{A} is Ricci flat, there exists a Ricci flat connection ∇ on TV so $\mathcal{R}_0^\nabla = \mathcal{A}$.*

The Ricci tensor ρ defines a natural short exact sequence which is equivariant with respect to the natural action of $\mathrm{GL}(V)$:

$$(1.c) \quad 0 \rightarrow \ker(\rho) \rightarrow \mathfrak{A}(V) \rightarrow V^* \otimes V^* \rightarrow 0.$$

If $\Theta \in V^* \otimes V^*$, set

$$(1.d) \quad H(\Theta)(x, y)z := \Theta(x, y)z - \Theta(y, x)z + \Theta(x, z)y - \Theta(y, z)x.$$

Clearly $H(\Theta)(x, y) = -H(\Theta)(y, x)$. One verifies that the Bianchi identity is satisfied and thus $H(\Theta) \in \mathfrak{A}(V)$ by computing:

$$\begin{aligned} & H(\Theta)(x, y)z + H(\Theta)(y, z)x + H(\Theta)(z, x)y \\ &= \Theta(x, y)z - \Theta(y, x)z + \Theta(x, z)y - \Theta(y, z)x \\ &+ \Theta(y, z)x - \Theta(z, y)x + \Theta(y, x)z - \Theta(z, x)y \\ &+ \Theta(z, x)y - \Theta(x, z)y + \Theta(z, y)x - \Theta(x, y)z \\ &= 0. \end{aligned}$$

Let $\{e_i\}$ be a basis for V . Let $\{e^i\}$ be the corresponding dual basis for V^* . Then:

$$\begin{aligned} & \rho(H(\Theta))(y, z) \\ &= e^i \{ \Theta(e_i, y)z - \Theta(y, e_i)z + \Theta(e_i, z)y - \Theta(y, z)e_i \} \\ &= \Theta(z, y) - \Theta(y, z) + \Theta(y, z) - m\Theta(y, z) \\ &= \frac{1-m}{2} \{ \Theta(z, y) + \Theta(y, z) \} + \frac{1+m}{2} \{ \Theta(z, y) - \Theta(y, z) \} \\ &= (1-m)\Theta_s(y, z) - (1+m)\Theta_a(y, z). \end{aligned}$$

So modulo a suitable renormalization, H splits the short exact sequence of Equation (1.c). Let $\mathfrak{W}(V) := \ker(\rho) \subset \mathfrak{A}(V)$ be the space of *Weyl projective curvature operators*. Let $\mathcal{P}(\mathcal{A})$ be the projection of \mathcal{A} on $\mathfrak{W}(V)$;

$$(1.e) \quad \mathcal{P}(\mathcal{A}) = \mathcal{A} + \frac{1}{m-1}H(\rho_s(\mathcal{A})) + \frac{1}{1+m}H(\rho_a(\mathcal{A})).$$

Following [20] one says that $\mathcal{A} \in \mathfrak{A}(V)$ is *projectively flat* if $\mathcal{P}(\mathcal{A}) = 0$ or, equivalently, if there exists $\Theta \in V^* \otimes V^*$ so $\mathcal{A} = H(\Theta)$. One says that a ∇ is projectively flat the associated curvature operator \mathcal{R}_P^∇ is projectively flat for all points P of M . Note that:

$$\begin{aligned} \dim\{S^2(V^*)\} &= \frac{1}{2}m(m+1), \quad \dim\{\mathfrak{W}(V)\} = \frac{m^2(m^2-4)}{3}, \\ \dim\{\Lambda^2(V^*)\} &= \frac{1}{2}m(m-1). \end{aligned}$$

One has the following result of Bokan [2] and Strichartz [22]; see also related work of Singer and Thorpe [21] in the Riemannian setting:

Theorem 1.5. *There is a $\mathrm{GL}(V)$ equivariant decomposition of $\mathfrak{A}(V)$ into irreducible $\mathrm{GL}(V)$ modules $\mathfrak{A}(V) = \mathfrak{W}(V) \oplus S^2(V^*) \oplus \Lambda^2(V^*)$.*

We shall omit the proof of the following result as it plays no role in our analysis and is only included for the sake of completeness; see [20] for further details:

Theorem 1.6. *Let ∇ and $\bar{\nabla}$ be torsion free connections. The following conditions are equivalent and define the notion of projective equivalence:*

- (1) $\mathcal{P}(\mathcal{R}^\nabla) = \mathcal{P}(\mathcal{R}^{\bar{\nabla}})$.
- (2) *There is a 1-form θ so $\nabla_x y - \bar{\nabla}_x y = \theta(x)y + \theta(y)x$.*
- (3) *The unparametrized geodesics of ∇ and of $\bar{\nabla}$ coincide.*

Theorem 1.5 gives rise to additional geometric representability questions. We will establish the following result in Section 5:

Theorem 1.7. *Let $\mathcal{A} \in \mathfrak{A}(V)$.*

- (1) *If \mathcal{A} is projectively flat, then there exists a projectively flat connection ∇ on TV so that $\mathcal{R}_0^\nabla = \mathcal{A}$.*
- (2) *If \mathcal{A} is projectively flat and Ricci symmetric, then there exists a projectively flat and Ricci symmetric connection ∇ on TV so that $\mathcal{R}_0^\nabla = \mathcal{A}$.*

The geometrical realization theorems discussed previously are equivariant with respect to the natural action of the general linear group $\mathrm{GL}(V)$. The decomposition of $\mathfrak{A}(V)$ as a $\mathrm{GL}(V)$ module has 3 components so there are 8 natural geometric

realization questions which are $\text{GL}(V)$ equivariant. Since the flat connection realizes the 0 curvature operator, there is only one natural GL equivariant geometric realization question which is not covered by the forgoing results. It is answered, in the negative, by the following result which we will establish in Section 6:

Theorem 1.8. *If ∇ is a projectively flat, Ricci antisymmetric, torsion free connection, then ∇ is flat. Thus if $0 \neq \mathcal{A} \in \mathfrak{A}(V)$ is projectively flat and Ricci antisymmetric, then \mathcal{A} is not geometrically realizable by a projectively flat, Ricci antisymmetric, torsion free connection.*

The geometric representability theorems of this paper can be summarized in the following table; the non-zero components of \mathcal{A} are indicated by \star .

$\mathfrak{W}(V)$	$S^2(V^*)$	$\Lambda^2(V^*)$		$\mathfrak{W}(V)$	$S^2(V^*)$	$\Lambda^2(V^*)$	
\star	\star	\star	yes	0	\star	\star	yes
\star	\star	0	yes	0	\star	0	yes
\star	0	\star	yes	0	0	\star	no
\star	0	0	yes	0	0	0	yes

2. THE PROOF OF THEOREM 1.1

Fix a basis $\{e_i\}$ for V . If $\mathcal{A} \in \mathfrak{A}(V)$, expand $\mathcal{A}(e_i, e_j)e_k = A_{ijk}^\ell e_\ell$. Theorem 1.1 will follow from the following observation:

Lemma 2.1. *Let $\Gamma_{uv}^l := \frac{1}{3}(A_{wuv}^l + A_{wvu}^l)x^w$ be the Christoffel symbols of a connection ∇ . Then ∇ is torsion free and $\mathcal{R}_0^\nabla(\partial_{x_i}, \partial_{x_j})\partial_{x_k} = A_{ijk}^l \partial_{x_l}$.*

Proof. Clearly $\Gamma_{uv}^\ell = \Gamma_{vu}^\ell$ so ∇ is torsion free. As Γ vanishes at the origin, we may use the curvature symmetries to compute:

$$\begin{aligned}
\mathcal{R}_0^\nabla(\partial_{x_i}, \partial_{x_j})\partial_{x_k} &= \{\partial_{x_i}\Gamma_{jk}^l - \partial_{x_j}\Gamma_{ik}^l\}(0)\partial_{x_l} \\
&= \frac{1}{3}\{A_{ijk}^l + A_{ikj}^l - A_{jik}^l - A_{jki}^l\}\partial_{x_l} \\
&= \frac{1}{3}\{A_{ijk}^l - A_{kij}^l + A_{ijk}^l - A_{jki}^l\}\partial_{x_l} \\
&= A_{ijk}^l \partial_{x_l}. \quad \square
\end{aligned}$$

3. THE PROOF OF THEOREM 1.3

Proof. We have by the first Bianchi identity of Equation (1.b) that

$$\text{Tr}\{\mathcal{R}(x, y)\} - \rho(y, x) + \rho(x, y) = 0.$$

This shows that Assertions (2) and (3) of Theorem 1.3 are equivalent. Note

$$\begin{aligned}
\mathcal{R}_{ijk}^l \partial_{x_\ell} &= \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k} \\
&= \{\partial_{x_i}\Gamma_{jk}^\ell - \partial_{x_j}\Gamma_{ik}^\ell + \Gamma_{in}^\ell \Gamma_{jk}^n - \Gamma_{jn}^\ell \Gamma_{ik}^n\}\partial_{x_\ell}, \\
\text{Tr}\{\mathcal{R}_{ij}\}dx_i \wedge dx_j &= \{\partial_{x_i}\Gamma_{jk}^k - \partial_{x_j}\Gamma_{ik}^k + \Gamma_{in}^k \Gamma_{jk}^n - \Gamma_{jn}^k \Gamma_{ik}^n\}dx_i \wedge dx_j \\
&= \{\partial_{x_i}\Gamma_{jk}^k - \partial_{x_j}\Gamma_{ik}^k\}dx_i \wedge dx_j = 2d\{\Gamma_{ij}^j dx_i\}.
\end{aligned}$$

Thus Assertions (1) and (2) of Theorem 1.3 are equivalent. Finally, we compute:

$$\nabla_{\partial_{x_i}} \{e^\Phi dx_1 \wedge \dots \wedge dx_m\} = \{\partial_{x_i}\Phi - \sum_k \Gamma_{ik}^k\} \{e^\Phi dx_1 \wedge \dots \wedge dx_m\}.$$

Thus there exists a parallel volume form on $\mathcal{O} \Leftrightarrow \Gamma_{ik}^k dx_i$ is exact. As every closed 1-form is locally exact, Assertions (1) and (4) of Theorem 1.3 are equivalent. \square

4. THE PROOF OF THEOREM 1.4

We extend the discussion in [7]. Fix a basis $\{e_1, \dots, e_m\}$ for V to identify V with \mathbb{R}^m ; let $x = (x_1, \dots, x_m)$ be the induced system of coordinates on V . Since any neighborhood of $0 \in V$ contains an open subset which is real analytically diffeomorphic to all of V , Theorem 1.4 will follow from the following result which is of interest in its own right:

Theorem 4.1. *Let $\mathcal{A} \in \mathfrak{A}(V)$. There exists a torsion free real analytic connection ∇ defined on an open neighborhood \mathcal{O} of $0 \in V$ so that $\mathcal{R}_0^\nabla = \mathcal{A}$ and so that $\rho(\mathcal{R}_P^\nabla)(\partial_{x_i}, \partial_{x_j}) = \rho(\mathcal{A})(e_i, e_j)$ for all $P \in \mathcal{O}$.*

The remainder of this section is devoted to the proof of Theorem 1.4. We complexify and set $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^m$. Let

$$|z| := (|z_1|^2 + \dots + |z_m|^2)^{1/2} \quad \text{and} \quad B_\delta := \{z \in \mathbb{C}^m : |z| < \delta\}$$

be the Euclidean length of $z \in \mathbb{C}^m$ and the open ball of radius $\delta > 0$ about the origin, respectively. Let \mathcal{H}_δ be the ring of all holomorphic functions q on B_δ such that $q(x)$ is real for $x \in \mathbb{R}^m \subset \mathbb{C}^m$. For $\nu = 0, 1, 2, \dots$ and for $q \in \mathcal{H}_\delta$, set

$$\|q\|_{\delta, \nu} := \sup_{0 < |z| < \delta} |q(z)| \cdot |z|^{-\nu}$$

where, of course, $\|q\| = \infty$ is possible. Let $\mathcal{H}(\delta, \nu) := \{q \in \mathcal{H}_\delta : \|q\|_{\delta, \nu} < \infty\}$; $(\mathcal{H}(\delta, \nu), \|\cdot\|_{\delta, \nu})$ is a Banach space. Clearly if $q \in \mathcal{H}(\delta, \nu)$, then we have the estimate

$$|q(z)| \leq \|q\|_{\delta, \nu} \cdot |z|^\nu \quad \text{for all } z \in B_\delta.$$

It is immediate that $\oplus_\nu \mathcal{H}(\delta, \nu)$ is a graded ring, i.e.

$$\mathcal{H}(\delta, \nu) \cdot \mathcal{H}(\delta, \mu) \subset \mathcal{H}(\delta, \mu + \nu).$$

If W is an auxiliary real vector space, we let $\mathcal{H}(\delta, \nu, W) := W \otimes_{\mathbb{R}} \mathcal{H}(\delta, \nu)$ be the appropriate function space of holomorphic functions taking values in W ; of particular interest will be the function spaces $\mathcal{H}(\delta, \nu, S^2(V^*))$ and $\mathcal{H}(\delta, \nu, \mathfrak{A}(V))$. Given a basis $\{f_i\}$ for W and given $q \in \mathcal{H}(\delta, \nu, W)$, we expand $q = \sum_i q_i f_i$ for $q_i \in \mathcal{H}(\delta, \nu)$ and define a Banach norm on $\mathcal{H}(\delta, \nu, W)$, by setting

$$\|q\|_{\delta, \nu} := \sup_i \|q_i\|_{\delta, \nu}.$$

Changing the basis for W yields an equivalent norm.

We shall use the canonical coordinate frame to identify $S^2(T^*V)$ with $V \times S^2(V^*)$ henceforth. The proof of Theorem 4.1 will be based on the following technical Lemma:

Lemma 4.2. *Let $\mathcal{A} \in \mathfrak{A}(\mathbb{R}^m)$. There exists $\delta = \delta(\mathcal{A}) > 0$, $C = C(\mathcal{A}) > 0$, and a sequence $\Gamma_\nu \in \mathcal{H}(\delta, 2\nu - 1, S^2(V^*) \otimes V)$ for $\nu = 1, 2, \dots$ so that:*

- (1) $\Gamma_{1, uv}{}^l := \frac{1}{3}(A_{wuv}{}^l + A_{wvu}{}^l)x^w$.
- (2) $\|\Gamma_\nu\|_{\delta, 2\nu-1} \leq C^{2\nu-1}$.
- (3) $\Gamma_{\nu, ij}{}^j = 0$ for $\nu \geq 2$.
- (4) If ∇_ν has Christoffel symbol $\Gamma_1 + \dots + \Gamma_\nu$, then $\|\rho_s(\mathcal{R}^{\nabla_\nu}) - \rho_s(\mathcal{A})\|_{\delta, 2\nu} \leq C^{2\nu}$.

We suppose for the moment such a sequence has been constructed. Choose $\delta_1 < \delta$ so $C^2\delta_1 < 1$. We set $\Gamma := \Gamma_1 + \Gamma_2 + \dots$. By Assertion (2), this series converges uniformly for $z \in B_{\delta_1}$. Thus the associated connection ∇ is a real torsion free connection on the real ball of radius δ_1 in V . Since uniform convergence in the holomorphic context implies the uniform convergence on compact subsets of all derivatives, Γ is a real analytic connection near $0 \in V$ with $R^\nabla = \lim_{\nu \rightarrow \infty} R^{\nabla_\nu}$.

Since $\Gamma_\nu = \Gamma_1 + O(|x|^3)$, we apply Lemma 2.1 to see $\mathcal{R}_0^{\nabla_\nu} = \mathcal{R}_0^{\nabla_1} = \mathcal{A}$. Define $\mathcal{L}(\Gamma_\nu)$ and $\Gamma_\nu \star \Gamma_\mu$ by setting:

$$\begin{aligned}\mathcal{L}(\Gamma_\nu)_{ijk}^l &:= \partial_{z_i} \Gamma_{\nu,jk}^l - \partial_{z_j} \Gamma_{\nu,ik}^l, \\ \{\Gamma_\mu \star \Gamma_\nu\}_{ijk}^\ell &:= \Gamma_{\mu,in}^\ell \Gamma_{\nu,jk}^n + \Gamma_{\nu,in}^\ell \Gamma_{\mu,jk}^n - \Gamma_{\mu,jn}^\ell \Gamma_{\nu,ik}^n - \Gamma_{\nu,jn}^\ell \Gamma_{\mu,ik}^n.\end{aligned}$$

We then have

$$\begin{aligned}(4.a) \quad \mathcal{R}^{\nabla_\nu} &= \sum_{\mu \leq \nu} \mathcal{L}(\Gamma_\mu) + \frac{1}{2} \left\{ \sum_{1 \leq \mu_1 \leq \nu} \Gamma_{\mu_1} \right\} \star \left\{ \sum_{1 \leq \mu_2 \leq \nu} \Gamma_{\mu_2} \right\} \\ &= \mathcal{R}^{\nabla_{\nu-1}} + \mathcal{L}(\Gamma_\nu) + \left\{ \sum_{1 \leq \mu_1 \leq \nu} \Gamma_{\mu_1} \right\} \star \Gamma_\nu - \frac{1}{2} \Gamma_\nu \star \Gamma_\nu, \\ \{\rho(\mathcal{L}(\Gamma_\nu))\}_{jk}^i &:= \partial_{x_i} \Gamma_{\nu,jk}^i - \partial_{x_j} \Gamma_{\nu,ik}^i.\end{aligned}$$

It is immediate from the definition that:

$$\begin{aligned}\rho(\Gamma_\mu \star \Gamma_\nu)_{jk} &= \Gamma_{\mu,\ell n}^\ell \Gamma_{\nu,jk}^n + \Gamma_{\nu,\ell n}^\ell \Gamma_{\mu,jk}^n - \Gamma_{\mu,jn}^\ell \Gamma_{\nu,\ell k}^n - \Gamma_{\nu,jn}^\ell \Gamma_{\mu,\ell k}^n \\ &= \rho(\Gamma_\mu \star \Gamma_\nu)_{kj}.\end{aligned}$$

Furthermore, if $\nu \geq 2$, then Assertion (3) yields that $\Gamma_{\nu,ik}^i = 0$ and thus $(\rho(\mathcal{L}(\Gamma_\nu)))$ is symmetric as well. Consequently $\rho_a(\mathcal{R}^{\nabla_\nu}) = \rho_a(\mathcal{L}(\Gamma_1)) = \rho_a(\mathcal{A})$. Thus by Assertion (4), we have

$$\begin{aligned}|\{\rho(\mathcal{R}^{\nabla_\nu})(z) - \rho(\mathcal{A})\}_{ij}| &\leq \lim_{\nu \rightarrow \infty} \|\rho(\mathcal{R}^{\nabla_\nu}) - \rho(\mathcal{A})\|_{\delta_1, 2\nu} \cdot |z|^{2\nu} \\ &\leq \lim_{\nu \rightarrow \infty} \|\rho_s(\mathcal{R}^{\nabla_\nu}) - \rho_s(\mathcal{A})\|_{\delta_1, 2\nu} \cdot |z|^{2\nu} = 0.\end{aligned}$$

Thus $\rho(\mathcal{R}^{\nabla_\nu}) = \rho(\mathcal{A})$ as desired and the proof of Theorem 1.4 will be complete once Lemma 4.2 is established.

Before establishing Lemma 4.2, we must establish the following solvability result:

Lemma 4.3. *If $\Theta \in \mathcal{H}(\delta, \nu, S^2(V^*))$, there exists $\Gamma \in \mathcal{H}(\delta, \nu + 1, S^2(V^*) \otimes V)$ so $\rho(\mathcal{L}(\Gamma)) = \Theta$, so $\|\Gamma\|_{\delta, \nu+1} \leq \|\Theta\|_{\delta, \nu}$, and so $\Gamma_{ij}^j = 0$.*

Proof. We have assumed throughout that $m \geq 3$. For each pair of indices $\{i, j\}$, not necessarily distinct, choose $k_{ij} = k_{ji}$ distinct from i and from j . Define the indefinite integral

$$(\int_k \Theta)(z) := z_k \int_0^1 \Theta(z_1, \dots, z_{k-1}, tz_k, z_{k+1}, \dots, z_m) dt.$$

Let $\Gamma_{ij}^\ell := \delta_{k_{ij}}^\ell \int_{k_{ij}} \Theta_{ij}$. It is immediate that $\|\Gamma\|_{\delta, \nu+1} \leq \|\Theta\|_{\delta, \nu}$. Since k_{ij} is distinct from i and j , we have that $\Gamma_{ij}^j = 0$. Furthermore, $\Gamma(x)$ is real if x is real. Finally, we use Equation (4.a) to complete the proof by checking:

$$(\rho(\mathcal{L}(\Gamma)))_{jk} = \partial_{x_i} \Gamma_{jk}^i = \Theta_{jk}. \quad \square$$

Proof of Lemma 4.2. Let $\mathcal{A} \in \mathfrak{A}(V)$. The Christoffel symbols Γ_1 are as in Lemma 2.1. Since Γ_1 is a homogeneous linear polynomial, there is a constant $C_1 > 0$ so

$$\|\Gamma_1\|_{\delta, 1} \leq C_1 \quad \text{and} \quad \|\rho(\mathcal{R}^{\nabla_1}) - \rho(\mathcal{A})\|_{\delta, 2} = \|\rho(\Gamma_1 \star \Gamma_1)\|_{\delta, 2} \leq C_1^2$$

for any $\delta > 0$. Let $C := 8m^2 C_1 > 0$. Choose $\delta > 0$ so that

$$(4.b) \quad C\delta^2 < 1 \quad \text{and} \quad \frac{C^3 \delta^2}{1 - C^2 \delta^2} \leq C_1.$$

If $\nu = 1$, Assertion (3) of Lemma 4.2 holds vacuously and Assertions (2) and (4) hold since $C > C_1$. Thus we may proceed by induction to establish Assertions (2), (3), and (4). We assume $\Gamma_1, \dots, \Gamma_\nu$ have been chosen with the desired properties. Use Lemma 4.3 to choose $\Gamma_{\nu+1} \in \mathcal{H}(\delta, 2\nu + 1, S^2(V^*) \otimes V)$ so that

$$\rho_s(\mathcal{L}(\Gamma_{\nu+1})) = -\rho_s(\mathcal{R}^{\nabla_\nu}) + \rho_s(\mathcal{A}) \quad \text{and} \quad \|\Gamma_{\nu+1}\|_{\delta, 2\nu+1} \leq C^{2\nu+1}.$$

We have the estimate:

$$(4.c) \quad \|\rho_s(\Gamma_\mu \star \Gamma_{\nu+1})\|_{\delta, 2\mu+2\nu} \leq 4m^2 \|\Gamma_\mu\|_{\delta, 2\mu-1} \cdot \|\Gamma_\nu\|_{\delta, 2\nu+1}.$$

As $\rho_s(\mathcal{R}^{\nabla_\nu} + \mathcal{L}(\Gamma_{\nu+1}))(z) = \rho_s(\mathcal{A})$ for any $z \in B_\delta$, Equations (4.a) and (4.c) yield

$$\begin{aligned}
& |\{\rho_s(\mathcal{R}^{\nabla_{\nu+1}})(z) - \rho_s(\mathcal{A})\}_{ij}| \\
&= |\{\rho_s[\mathcal{R}^{\nabla_\nu} + \mathcal{L}(\Gamma_{\nu+1}) + \sum_{\mu \leq \nu} \Gamma_\mu \star \Gamma_{\nu+1} + \frac{1}{2}\Gamma_{\nu+1}\Gamma_{\nu+1}](z) - \rho_s(\mathcal{A})\}_{ij}| \\
&= |\{\rho_s[\sum_{\mu \leq \nu} \Gamma_\mu \star \Gamma_{\nu+1} + \frac{1}{2}\Gamma_{\nu+1}\Gamma_{\nu+1}](z)\}_{ij}| \\
&\leq 4m^2\{C_1|z| + C^3|z|^3 + \dots + C^{2\nu+1}|z|^{2\nu+1}\}C^{2\nu+1}|z|^{2\nu+1} \\
&\leq 4m^2\{C_1 + C^3\delta^2 + \dots + C^{2\nu+1}\delta^{2\nu}\}C^{2\nu+1}|z|^{2\nu+2}.
\end{aligned}$$

Estimating using a geometric series and applying Equation (4.b) completes the inductive step by showing

$$\begin{aligned}
|\{\rho_s(\mathcal{R}^{\nabla_{\nu+1}})(z) - \rho_s(\mathcal{A})\}_{ij}| &\leq 4m^2 \left\{ C_1 + \frac{C^3\delta^2}{1 - C^2\delta^2} \right\} C^{2\nu+1}|z|^{2\nu+2} \\
&\leq 8m^2 C_1 C^{2\nu+1}|z|^{2\nu+2} \leq C^{2\nu+2}|z|^{2\nu+2}.
\end{aligned}$$

The proof of Lemma 4.2 and thereby of Theorem 4.1 and thus of Theorem 1.4 is now complete. \square

5. THE PROOF OF THEOREM 1.7

Proof. Let $\{e_i\}$ be a basis for V and let $\{x_i\}$ be the associated coordinate system on V . Let θ be a 1-form. Motivated by Theorem 1.6, we define a connection ∇^θ so

$$\nabla_x^\theta y = \theta(x)y + \theta(y)x$$

if x and y are coordinate vector fields. Set $\Psi(x, y) = x\theta(y)$ and let H be as in Equation (1.d). Then:

$$\begin{aligned}
\mathcal{R}^{\nabla^\theta}(x, y)z &= \theta(x)\theta(y)z + \theta(x)\theta(z)y - \theta(y)\theta(x)z - \theta(y)\theta(z)x \\
&\quad + x(\theta(y))z + x(\theta(z))y - y(\theta(x))z - y(\theta(z))x \\
&= H(\theta \otimes \theta + \Psi).
\end{aligned}$$

Consequently ∇^θ is projectively flat. Let $\Theta \in V^* \otimes V^*$. Set $\theta = x_i \Theta_{ij} dx_j$. Then $\theta(0) = 0$ and $\Psi(0) = \Theta$ so

$$\rho(\mathcal{R}^{\nabla^\theta})(0) = (1 - m)\Theta_s - (m + 1)\Theta_a.$$

Consequently, given any $\mathcal{A} \in \mathfrak{A}(V)$ with $\mathcal{P}(\mathcal{A}) = 0$ there exists a torsion free projectively flat connection ∇^θ so that $\mathcal{R}_0^{\nabla^\theta} = \mathcal{A}$. This proves Theorem 1.7 (1). Furthermore, if Θ is symmetric, then $d\theta = 0$. Thus Ψ is symmetric for any point $P \in V$ and $\mathcal{R}^{\nabla^\theta}$ is Ricci symmetric. This proves Theorem 1.7 (2). \square

6. THE PROOF OF THEOREM 1.8

Proof. Let $\mathcal{R}^\nabla(x, y; z) = (\nabla_z \mathcal{R}^\nabla)(x, y)$ be the covariant derivative of the curvature; we then have the *second Bianchi identity*:

$$(6.a) \quad 0 = \mathcal{R}^\nabla(x, y; z) + \mathcal{R}^\nabla(y, z; x) + \mathcal{R}^\nabla(z, x; y).$$

Suppose $\rho(\mathcal{R}^\nabla) \in \Lambda^2(V^*)$. Let $\omega_{ij} := -\frac{1}{m+1}\rho(\partial_{x_i}, \partial_{x_j})$. By Equations (1.d) and (1.e),

$$\mathcal{R}^\nabla(\partial_{x_i}, \partial_{x_j})\partial_{x_k} = 2\omega_{ij}\partial_{x_k} + \omega_{ik}\partial_{x_j} - \omega_{jk}\partial_{x_i}.$$

Covariantly differentiating this relation yields:

$$\begin{aligned}
\mathcal{R}^\nabla(\partial_{x_i}, \partial_{x_j}; \partial_{x_\ell})\partial_{x_k} &= 2\omega_{ij;\ell}\partial_{x_k} + \omega_{ik;\ell}\partial_{x_j} - \omega_{jk;\ell}\partial_{x_i}, \\
\mathcal{R}^\nabla(\partial_{x_j}, \partial_{x_\ell}; \partial_{x_i})\partial_{x_k} &= 2\omega_{j\ell;i}\partial_{x_k} + \omega_{jk;i}\partial_{x_\ell} - \omega_{\ell k;i}\partial_{x_j}, \\
\mathcal{R}^\nabla(\partial_{x_\ell}, \partial_{x_i}; \partial_{x_j})\partial_{x_k} &= 2\omega_{\ell i;j}\partial_{x_k} + \omega_{\ell k;j}\partial_{x_i} - \omega_{ik;j}\partial_{x_\ell}.
\end{aligned}$$

Summing and applying the second Bianchi identity of Equation (6.a) yields

$$(6.b) \quad 0 = (2\omega_{ij;\ell} + 2\omega_{j\ell;i} + 2\omega_{\ell i;j})\partial_{x_k} \\ + (\omega_{ik;\ell} - \omega_{\ell k;i})\partial_{x_j} + (\omega_{\ell k;j} - \omega_{jk;\ell})\partial_{x_i} + (\omega_{jk;i} - \omega_{ik;j})\partial_{x_\ell}.$$

Let $\{i, j, \ell\}$ be distinct indices. Set $k = i$. Examining the coefficient of ∂_{x_j} in Equation (6.b) yields

$$0 = \omega_{ii;\ell} - \omega_{\ell i;i} = \omega_{i\ell;i}.$$

Polarizing this identity then yields

$$\omega_{i\ell;j} + \omega_{j\ell;i} = 0 \quad \text{and} \quad \omega_{\ell i;j} + \omega_{\ell j;i} = 0.$$

Next we set $k = \ell$ and examine the coefficient of ∂_{x_k} in Equation (6.b) to see

$$0 = 2\omega_{ij;k} + 2\omega_{jk;i} + 2\omega_{ki;j} + \omega_{jk;i} - \omega_{ik;j} \\ = 2\omega_{ij;k} + 3\omega_{jk;i} + 3\omega_{ki;j} = -2\omega_{kj;i} + 3\omega_{jk;i} - 3\omega_{kj;i} = 8\omega_{jk;i}.$$

Thus if $\{x, y, z\}$ are linearly independent vectors, then $\nabla_x \omega(y, z) = 0$; since the set of all triples of linearly independent vectors is dense in the set of all triples, this relation holds by continuity for all $\{x, y, z\}$ and thus $\nabla \omega = 0$. We compute:

$$0 = \{(\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]})\omega\}(z, w) \\ = \omega(\mathcal{R}^\nabla(x, y)z, w) + \omega(z, \mathcal{R}^\nabla(x, y)w) \\ = 4\omega(x, y)\omega(z, w) + 2\omega(x, z)\omega(y, w) - 2\omega(x, w)\omega(y, z).$$

Set $x = z$ and $y = w$ to see that $6\omega(x, y)^2 = 0$. Consequently $\omega = 0$ so $\mathcal{R} = 0$. \square

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